Math 53: Multivariable Calculus **Sections 102, 103**

Worksheet for 2020-03-20

Conceptual questions

Question 1. For each of the following integrals, would you rather use $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$, $\mathrm{d}y\,\mathrm{d}x$, or $r\,\mathrm{d}r\,\mathrm{d}\theta$?

(a) $\iint_D e^{y^3}\,\mathrm{d}A$ where D is some region in the xy-plane.

(b) $\iint_D \mathrm{d}A$ where D is the region bounded by $y = x$ and $y^2 = 2x + 2$.

(c) $\iint_{x^2+y^2\le 1} e^{-x^2-y^2}$.

Computations

Problem 1 (Polar integration)**.** Consider the region between the paraboloid $z = x^2 + y^2$ and the plane $z = x + y$.

(a) By setting the two expressions of z equal, you obtain an equation in x and y that describes a curve in the xy-plane; the region in question resides over the area enclosed by this curve. It should be a familiar shape—draw it.

(b) Of the two surfaces, which one is the top of the 3D region in question and which one is the bottom?

(c) What polar curve encloses your region of integration? What are the θ bounds?

(d) Set up an integral for the region in question.

Problem 2 (Gaussian integral)**.** One of the most spectacular and unexpected applications of polar integration is to compute the following integral (of paramount importance in probability and statistics):

$$A = \int_{-\infty}^{\infty} e^{-x^2/2}\,\mathrm{d}x.$$

If we square this expression, we get

$$A^2 = \int_{-\infty}^{\infty} e^{-x^2/2}\,\mathrm{d}x \int_{-\infty}^{\infty} e^{-y^2/2}\,\mathrm{d}y$$

where I can use a different letter for the second integral because it's just a dummy variable anyway. But then I can combine these:

$$A^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2/2}e^{-y^2/2}\,\mathrm{d}y\,\mathrm{d}x$$

and here the integrand can be rewritten as a single exponential. Use this to compute A. (Switch to polar, and think about what your r and θ bounds ought to be.)